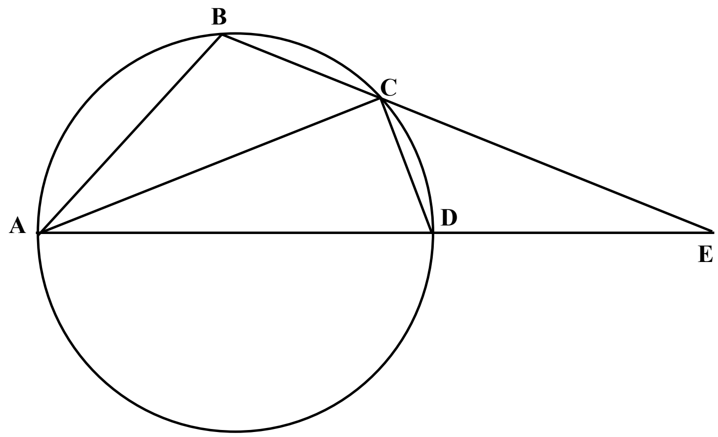
§11.1

Chapter Eleven

The ratio of other chords can be found with no less certainty, and for this purpose the following Lemma is useful.

LEMMA.

If the line AC bisects the angle in the periphery BAD; and AC, CE are equal; then DE, BA are equal also. For the angles BAC, CAD, CED are equal by hypothesis, and from *Prop. 5, Book 1*. And the angles ABC, CDE also are equal, by *Prop. 22, Book 3* and *Prop. 13, Book 1* . Therefore CDE, CBA are similar triangles; And with CA, CE equal *from the construction*, BA, DE will be equal.



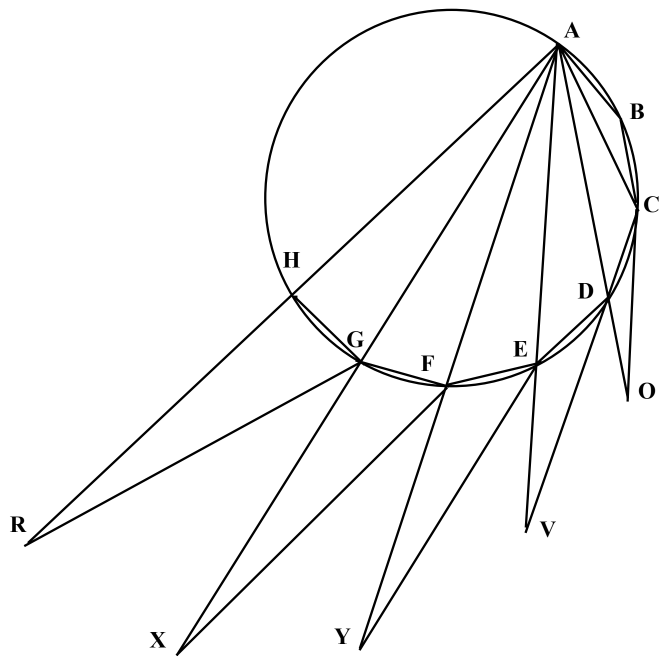
[Figure 11-1]

Let AB, BC, CD, DE, EF be equal chords inscribed in a circle : in like manner AC, CO; AD, DV; AE, EY; AF, FX; AG, GR are equal; then AB, DO; AC, EV; AD, FY; AE, GX; AF, HR are equal *by the preceding lemma.*

Let the first AB, which is written

thus: 1(1)
AC second 1(2) } *cont. proport.*
AO third 1(3)

AD will be the third, for which DO, equal to the first, is taken.



[Figure 11-2]

AB	1①
AC	1②
AO	1③
AD	1③ — AB 1①
AV	1④ — 1②
AE	1④ — 1② — AC 1②
AE	1④ — 2②
AY	1⑤ — 2③
AF	1⑤ — 2③ — AD
AF	1⑤ — 3③ + 1①
AX	1⑥ — 3④ + 1②
AG	1⑥ — 3④ + 1② — AE
AG	1⑥ — 4④ + 3②
AR	1⑦ — 4⑤ + 3③
AH	1⑦ — 4⑤ + 3③ — AF
AH	1⑦ — 5⑤ + 6③ — 1①

[Table 11-1]

With all of these triangles the ratio of the equal legs to their own base[1] *is the same .*

Proportionals

AB	1①
AC	1②
AD	1③ — 1①
AV	1④ — 1②

If AE, let it be 1④ — 2②: AY will be 1⑤ — 2③ , from which AD or FY is taken away: 1③ — 1①,

there remains AF, 1⑤ — 3③ + 1① .

If any lines are in continued proportion, of which the first is the chord subtended by some arc, the second truly is the chord of double the arc; the third, if the first is taken away, is the chord of the triple arc: the fourth, with double the second taken away, is the chord of the quadruple arc. The fifth with the first added on, with three of the third taken away, is the chord of the quintuple of the arc: the sixth with three of the second added on, and with four of the fourth taken away, is the chord of six times the arc. The seventh with six of the third added on and five of the fifth taken away with the first, is the chord of seven times the arc, etc., as you can see in the following table .

	—	+	—	+	—	+	—
First							
Second	—						
Third	1 (1)						
Fourth	2 (2)	+					
Fifth	3 (3)	1 (1)					
Sixth	4 (4)	3 (2)	—				
Seventh	5 (5)	6 (3)	1 (1)				
Eighth	6 (6)	10 (4)	4 (2)	+			
Ninth	7 (7)	15 (5)	10 (3)	1 (1)			
Tenth	8 (8)	21 (6)	20 (4)	5 (2)	—		
Eleventh	9 (9)	28 (7)	35 (5)	15 (3)	1 (1)		
Twelfth	10 (10)	36 (8)	56 (6)	35 (4)	6 (2)	+	
Thirteenth	11 (11)	45 (9)	84 (7)	70 (5)	21 (3)	1 (1)	
Fourteenth	12 (12)	55 (10)	120 (8)	126 (6)	56 (4)	7 (2)	—
Fifteenth	13 (13)	66 (11)	165 (9)	210 (7)	126 (5)	28 (3)	1 (1)
Sixteenth	14 (14)	78 (12)	220 (10)	330 (8)	252 (6)	84 (4)	8 (2)
A	B	C	D	E	F	G	H

[Table 11-2]

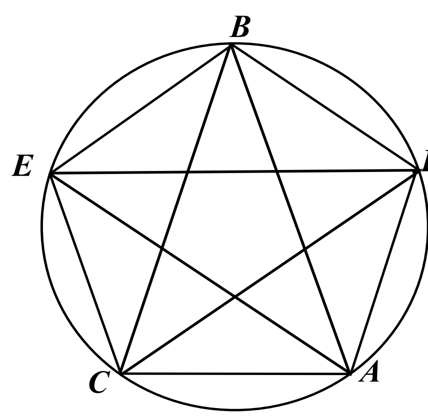
Column A gives the lines in continued proportion, with columns B,D,F,H, taken away; and C,E,G with the proportions added.

For the chord subtended by the arc times ten, it is the tenth proportional, increased by 21 of the sixth and 5 of the second, from which is taken 8 of the eighth and twenty of the fourth.

AB	190211303259	(1)	*5 Sides, subtending 2 angles*
AC	117557050458	(2)	*5 Sides, subtending 4 angles*
	72654252801	(3)	
	44902797657	(4)	*Numbers in continued*
	27751455144	(5)	*proportion*
	17151342513	(6)	
	10600112631	(7)	
	6551229882	(8)	
	4048882749	(9)	

1. AB *144 Degrees Chord*
2. AC *288 Degrees Chord*
3. AD 432 [=] 360 & 72 *Difference of 3rd & 1st*
4. AE 576 [=] 360 & 216 *Diff. 4th & two of 2nd*
5. A 720 [=] 360 & 360 *Diff. 5th + 1st, & 3 of 3rd*

[Table 11-3]



[Figure 11-3]

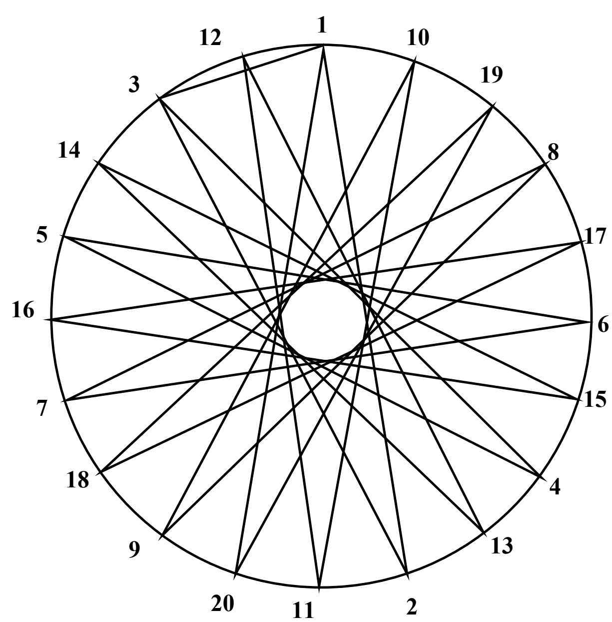
235114100916	2. *Second*
44902797657	1. *Fourth*
190211303259	AE
27751455144	1.*Fifth*
190211303259	1.*First*
217962758403	*Sum* 1⑤ + 1①
217962758403	*Three of the third. By taking* 3③ *0 remains.*
4048882749	1⑨
416271827160	15⑤
190211303259	1①
610532013168	1⑨ + 15⑤ + 1①
74200788417	7⑦
726542528010	10③
800743316427	7⑦ + 10③
190211303259	AE 7⑦ + 10③ — 1⑨ — 15⑤ — 1①

[Table 11-4]

[Note: Tables 11-3 and 11-4 have been badly affected by errors, that presumably have originated by the letters E and D occupying each other's positions in the original Figure 11-3: Accordingly, E and D have been interchanged in the diagram reproduced here from those in the original diagram, to give agreement with the tables, and to make mathematical sense.]

1.2. 1975376681190275	①	*Chord 162 Deg. 1.2*
1.3 618033988749897	②	*Chord 324 Deg. 1.3*
193363632813541	③	
60497472914844	④	
18927779623439	⑤	
5921914159584	⑥	
1852782947137	⑦	
579678218332	⑧	
181363303960	⑨	
56742942866	⑩	
17753103824	⑪	

Numbers in continued proportion
[Table 11-5]



[Figure 11-4]

	Numbers in continued proportion
1975376681190275	1① *Chord of 162 Degrees*
	1③ A
193363632813541	
1782013048376734	*Chord of 486, 126* 1① — 1③ *above a single circle.*
1236067977499794	2.2nd *Subtending 324 Degrees*
60497472914844	1.4th B
1175570504584950	*Chord of 648, 288* 2② — 1④ *above a single circle.*
580090898440623	3.3rd
1975376681190275	1.1st C
	1.5th
18927779623439	
1994304460813714	*Sum* 1⑤ + 1①
1414213562373091	*Chord* 810, 90 1⑤ + 1① — 3③ *above two circles.*
241989891659376	4.4th
1854101966249691	3.2nd D
5921914159584	1.6th
1860023880409275	*Sum* 1.6th + 3②
1618033988749899	*Chord*: 972, 252 1⑥ + 3② — 4④
197537	*First Chord* 162:0':
61803	*Second Chord 324:0': or 36:0':*

[Table 11-6]

If the multiple arc is left with a single circle or three circles, the contrary signs are assumed; as in the examples A and B. For they were the signs in the Table (from section three of this chapter) 1③ — 1① and 1④ — 2② ; but contrary to this 1① — 1③ and 2② — 1④ . If however there should remain two or four circles, then the signs are as in examples C and D.

§11.1 Notes on Chapter Eleven.

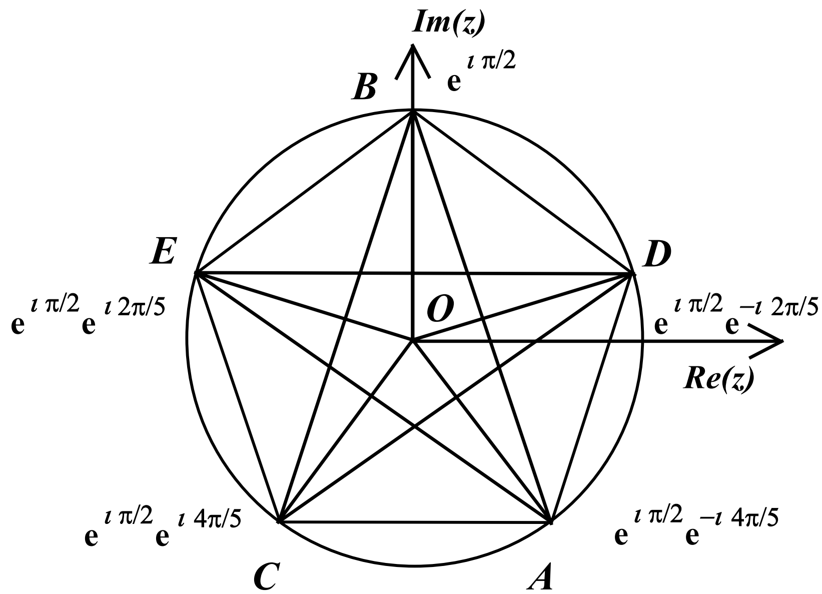
[1] Following Briggs' usual procedure with sets of similar isosceles triangles, if (a, a, pa) are the lengths of the sides of the first triangle ABC in Figure 11-2, where a is the original length of the equal side, and p is the common ratio used; the sides of the second triangle ACO constructed will be (pa, pa, p^2a), the third (p^2a, p^2a, p^3a), and so on. The length of the first chord AB = a = 1①; the second chord AC = pa = 1②; while the length of the third chord AD = AO - DO = AO - AB = $p^2a - a$ = 1③ — 1①. The lengths of the chords L_n can then be found from the iterative scheme: $L_n = pL_{n-1} - L_{n-2}$

$L_1 = a$; $L_2 = pa$; $L_3 = (p^2 - 1)a$; $L_4 = pL_3 - L_2 = p(p^2 - 1)a - pa = p((p^2 - 1) - 1)a = p^3a - 2pa$;

$L_5 = pL_4 - L_3 = p^2((p^2 - 1) - 1)a - (p^2 - 1)a = (p^4 - 3p^2 + 1)a$;

$L_6 = pL_5 - L_4 = p(p^2((p^2 - 1) - 1)a - (p^2 - 1)a) - p((p^2 - 1) - 1)a = (p^5 - 4p^3 + 3p)a, \ldots$, etc

Let us examine how the various regular figures fare according to this scheme:



I. The equilateral triangle: $p = 1$, as $L_1 = a$; $L_2 = pa = L_1$;

II. The square: $L_1 = a$; $L_2 = pa$; $L_3 = p^2a - a = L_1 = a$; Hence $p = \sqrt{2}$.

III. The pentagon: $L_4 = p^3a - 2pa = L_1 = a$; and $L_2 = pa = L_3 = p^2a - a$; hence, $p^2 = p + 1$: this has solution $\tau = (1 + \sqrt{5})/2$, (and $-1/\tau$).

IV. The hexagon: $L_5 = (p^4 - 3p^2 + 1)a = L_1 = a$; and $L_2 = L_4$, being symmetric about the central chord L_3. Hence, $pa = p^3a - 2pa$, or $p = \sqrt{3}$.

V. The heptagon: $L_1 = L_6$; $L_2 = L_5$; and $L_3 = L_4$. From the last equality, $p^2a - a = p^3a - 2pa$, or $p^3 - p^2 - 2p + 1 = 0$